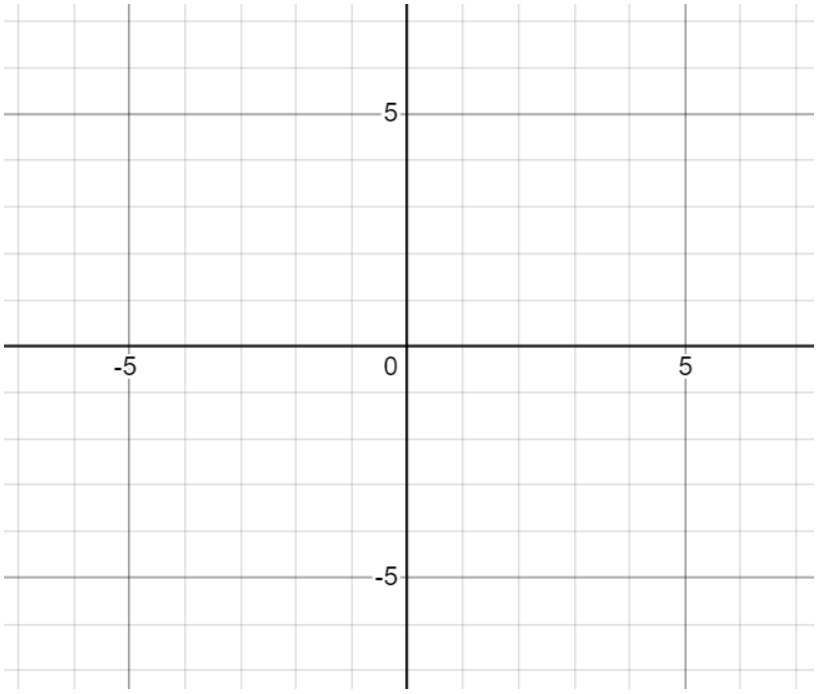
AP Calculus AB/BC - Differential Equations

Show all your work. Clearly indicate the methods you use, because you will be scored on the correctness of your methods as well as on the accuracy and completeness of your results and explanations. Use complete sentences for all explanations.

1. A colony of bacteria is growing in a lab. The rate of change is proportional to the population $P(t)$ of bacteria. Let t be time measured in hours. The colony has 100 bacteria at time $t = 0$ and 300 bacteria at time $t = 4$.
    a) Write a differential equation to model this relationship.

    b) Draw a slope field for the differential equation.



c) Solve your equation in (a) to find an expression for $P(t)$.

2. Consider the differential equation $\frac{dy}{dx} = \cos^2 y$ and the solution $y = f(x)$ with initial values $x = 0$ and $y = 0$.

   a) Find the equation of the tangent line to $f(x)$ at $x = 0$. Is $f(x)$ increasing, decreasing, or neither at $(0,0)$? Explain.

   b) Find the value of $\frac{d^2y}{dx^2}$ at $x = 0$. Is $f(x)$ concave up, concave down, or neither at $(0,0)$? Explain.

   c) Solve for $y = f(x)$.

d) For, $\frac{-\pi}{2} < x < \frac{\pi}{2}$, find an expression for

$$\frac{f(\tan x)}{x}$$

3. If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity $v(t)$ is modeled by the initial value problem

$$m\frac{dv}{dt} = mg - kv^2$$

$$v(0) = 0$$

where t represents time in seconds, g is the acceleration due to gravity, and k is a constant that depends on the body's aerodynamic properties and the density of the air. Assume that the fall is short enough so that variation in the air's density will not affect the outcome.

a) A technique of integration is called "partial fraction decomposition". Let

$$\frac{1}{1-y^2} = \frac{A}{1+y} + \frac{B}{1-y}$$

Find A and B, and use it to find the solution to the differential equation

$$\frac{dy}{dx} = 1 - y^2$$

b) Use this technique to solve the initial value problem $v(0) = 0$,

$$m\frac{dv}{dt} = mg - kv^2$$

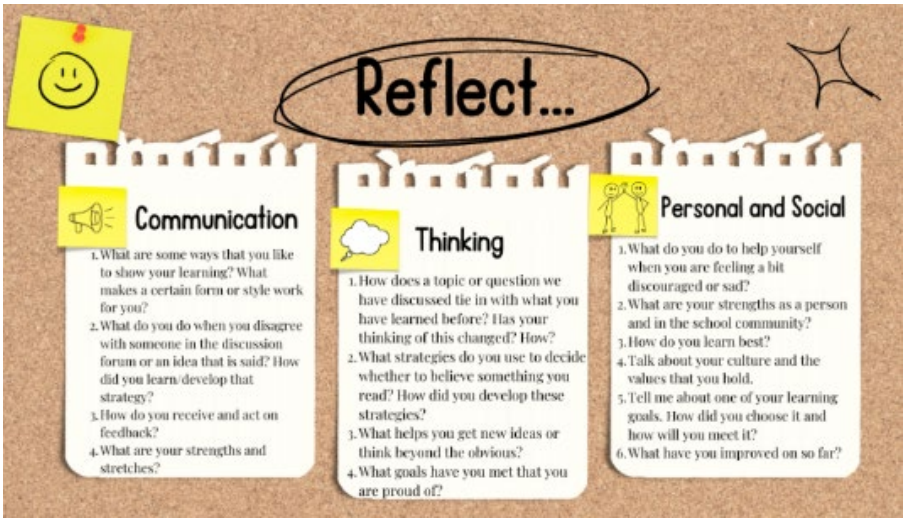
Try your best! The answer should be

$$v(t) = \sqrt{\frac{mg}{k}}\frac{e^{at} - e^{-at}}{e^{at} + e^{-at}} \text{ where } a = \sqrt{\frac{gk}{m}}$$

c) Interpret the meaning of $\lim_{t\to\infty} v(t)$.

d) For a 75kg skydiver (mg = 735) diving feet down first, with time in seconds and distance in m, a typical value for k is 0.075. What is the diver's limiting velocity in m/s?

4. Think about your learning journey in this unit. Pick a category and write a response to ONE of the category's questions in a few complete sentences.



~The End~

Quick checklist:

Did you…

- O State your final answer and explanations using full sentences?
- O Explain in a few sentences every time you have a Yes/No answer?
- O Use equal signs correctly where necessary?